Identification and Estimation of Local Treatmetn Effects through IV Estimation of Binary Outcome Structural Mean Models

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Binary outcome Y, treatment X and random assignment Z. Non-compliance.

Potential outcomes. $Y(x)$ is the counterfactual value of Y if X is set to x:

$$Y(1) = Y(x = 1)$$
$$Y(0) = Y(x = 0)$$

Equivalently for $X(z)$.

SMM

Non-parametric saturated additive SMM is defined by

$$E\left[Y(1) \mid X=1, Z\right]-E\left[Y(0) \mid X=1, Z\right]=\gamma(1, Z, \psi)=\psi_0+\psi_1 Z$$

or

$$E[Y \mid X, Z]-E\left[Y(0) \mid X, Z\right]=X\left(\psi_0+\psi_1 Z\right)$$

using conditional mean independence

$$E\left[Y(0) \mid Z=1\right]=E\left[Y(0) \mid Z=0\right]$$

it follows that

$$E\left[Y-X\left(\psi_0+\psi_1\right) \mid Z=1\right]=E\left[\left(Y-X \psi_0\right) \mid Z=0\right]$$

The nonparametric saturated multiplicative SMM is defined as

$$E\left[Y(1) \mid X = 1, Z\right] = E\left[Y(0) \mid X = 1, Z\right]\exp\left(\gamma(1, Z, \theta)\right)$$

or

$$E\left[Y \mid X, Z\right] = E\left[Y(0) \mid X, Z\right]\exp\left(X\left(\theta_0 + \theta_1 Z\right)\right)$$

Exclusion of Treatment in the Control Group

If $P(X=1|Z=0)=0$, then

$$E[Y|Z=0]=E\left[Y(0)\right].$$

As

$$\begin{aligned}E\left[Y(0)|Z=1\right]=&E\left[Y(0)|X=1,Z=1\right]P(X=1|Z=1)\\&+E\left[Y(0)|X=0,Z=1\right]P(X=0|Z=1)\end{aligned}$$

and $E\left[Y(0)|Z=1\right]=E\left[Y(0)|Z=0\right]$ we can construct an estimate for the counterfactual $E\left[Y(0)|X=1,Z=1\right]$.

Resulting in

$$\psi_0 + \psi_1 = \frac{E\left[Y \mid Z = 1\right] - E\left[Y \mid Z = 0\right]}{E\left[X \mid Z = 1\right]}$$

And

$$\exp\left(\theta_0 + \theta_1\right) = \frac{E\left[XY \mid Z = 1\right]}{E\left[Y \mid Z = 0\right] - E\left[\left(1 - X\right)Y \mid Z = 1\right]}$$

We can also identify the causal odds ratio of the treated with $Z=1$. If we define

$$\frac{E\left[Y(1)\mid X=1,Z\right]}{1-E\left[Y(1)\mid X=1,Z\right]}=\frac{E\left[Y(0)\mid X=1,Z\right]}{1-E\left[Y(0)\mid X=1,Z\right]}\exp\left(\xi_0+\xi_1 Z\right)$$

then

$$\exp\left(\xi_0+\xi_1\right)=\frac{E\left[Y\mid X=1,Z=1\right]}{1-E\left[Y\mid X=1,Z=1\right]}\Big/\frac{E\left[Y\mid Z=0\right]-E\left[\left(1-X\right)Y\mid Z=1\right]}{E\left[X\mid Z=1\right]-E\left[Y\mid Z=0\right]+E\left[\left(1-X\right)Y\mid Z=1\right]}$$

And this is the parameter estimated by the Vansteelandt and Goetghebeur (2003) method.

No Exclusion of Treatment in the Control Group

When there is also non-compliance in the control group, SMM cannot identify both parameters ψ_0 and ψ_1, or θ_0 and θ_1.
One assumption that can be made is that there is no effect modification by Z, and therefore $\psi_1 = 0$ and $\theta_1 = 0$.

In that case, the linear IV estimand

$$\psi_0 = \frac{E[Y \mid Z=1] - E[Y \mid Z=0]}{E[X \mid Z=1] - E[X \mid Z=0]}$$

estimates the causal average effect of treatment on the treated, and the causal relative risk parameter for the treated is

$$\exp(\theta_0) = 1 - \frac{E[Y \mid Z=1] - E[Y \mid Z=0]}{E[(1-X)Y \mid Z=1] - E[(1-X)Y \mid Z=0]}$$

In the absence of the no-effect modification assumption, but with a monotonicity assumption

$$X(1) \geq X(0)$$

Imbens and Angrist (1994) showed that the linear IV estimand estimates a local average treatment effect, LATE

$$\psi_0 = E\left[Y(1) - Y(0) \mid X(1) > X(0)\right]$$

And Angrist (2001) showed that

$$\exp(\theta_0) = \frac{E\left[Y(1) \mid X(1) > X(0)\right]}{E\left[Y(0) \mid X(1) > (0)\right]}$$

a local relative risk parameter.

By also performing a multiplicative SMM on $(1-Y)$, with resulting estimand $\exp(\theta_0^*)$, we get

$$\frac{\exp(\theta_0)}{\exp(\theta_0^*)} = \frac{E\left[Y(1)\,|\,X(1)>X(0)\right]}{1-E\left[Y(1)\,|\,X(1)>X(0)\right]} \Big/ \frac{E\left[Y(0)\,|\,X(1)>(0)\right]}{1-E\left[Y(0)\,|\,X(1)>X(0)\right]}$$

A local causal odds-ratio.

All this follows as Abadie (2002, 2003) has shown that

$$E\left[Y(1)\,|\,X(1)>X(0)\right] = \frac{E[YX \mid Z=1]-E[YX \mid Z=0]}{E[X \mid Z=1]-E[X \mid Z=0]}$$

$$E\left[Y(0)\,|\,X(1)>X(0)\right] = \frac{E\left[Y(1-X)\,|\,Z=1\right]-E\left[Y(1-X)\,|\,Z=0\right]}{E\left[(1-X)\,|\,Z=1\right]-E\left[(1-X)\,|\,Z=0\right]}.$$

A Structural Model

The no effect modification assumption for the SMMs can hold in the simple models

$$E\left[Y \mid X = x, U = u\right] = g\left(x\right) + u$$
$$E\left[Y \mid X = x, W = w\right] = h\left(x\right) w$$

and using moment conditions $E\left[Z\left(Y - g\left(x\right)\right)\right] = 0$ and $E\left[Z\frac{Y - h\left(x\right)}{h\left(x\right)}\right] = 0.$

For binary outcomes and treatment, a simple structural model is specified as

$$y = E\left[Y \mid X = x, U = u\right] = 1\left\{\alpha + \beta x - u > 0\right\}$$
$$x = E\left[X \mid Z = z, V = v\right] = 1\left\{\gamma + \delta z - v > 0\right\}$$

and

$$E\left[Y(1) \mid X = 1, Z = 1\right] = P\left(U < \alpha + \beta \mid V < \gamma + \delta\right) = \frac{F\left(\alpha + \beta, \gamma + \delta; \rho\right)}{F_v\left(\gamma + \delta\right)}$$
$$E\left[Y(0) \mid X = 1, Z = 1\right] = P\left(U < \alpha \mid V < \gamma + \delta\right)$$
$$E\left[Y(1) \mid X = 1, Z = 0\right] = P\left(U < \alpha + \beta \mid V < \gamma\right)$$
$$E\left[Y(0) \mid X = 1, Z = 0\right] = P\left(U < \alpha \mid V < \gamma\right)$$

and therefore the no effect modification assumption is violated, and the SMMs will not identify the causal effects on the treated.

However, as the monotonicity assumption is clearly valid in this model, the SSMs will estimate local effects.

In order to gauge the discrepancies between various causal parameters we calculate these assuming that

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

for various parameter values, setting $\alpha = 0, \beta = 0.1, P(Z = 1) = 0.5$.

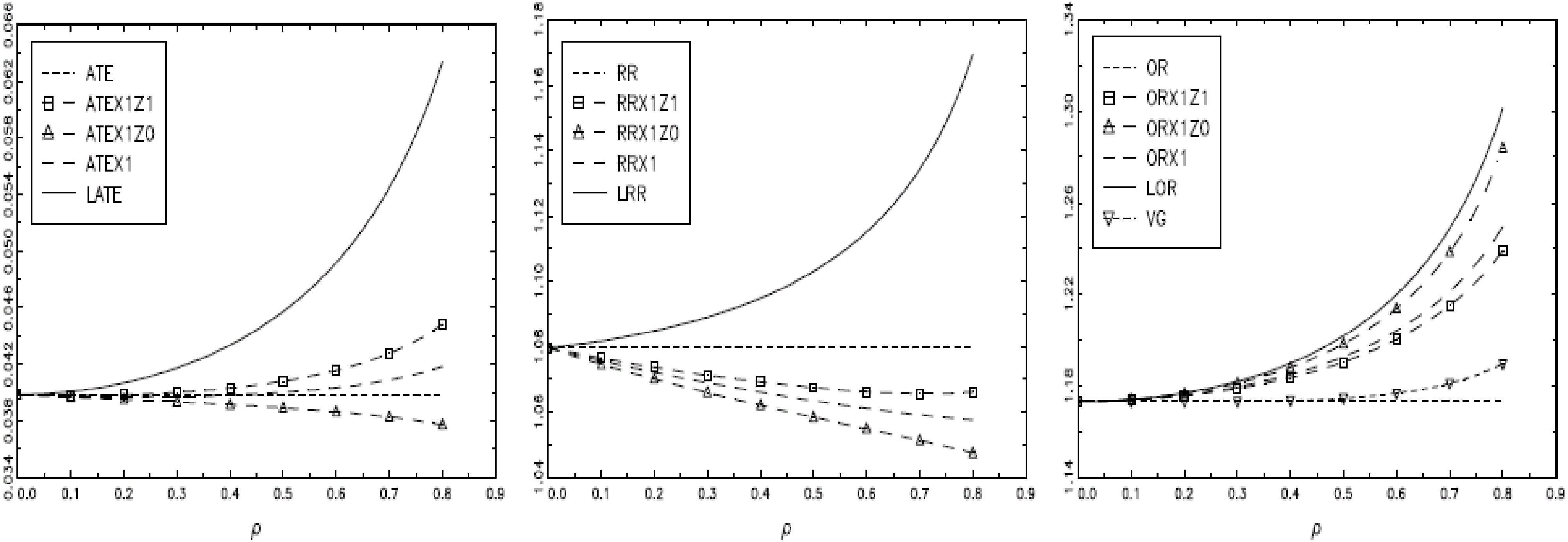

Figure 1. $\gamma = 0$, $\delta = 0.500$

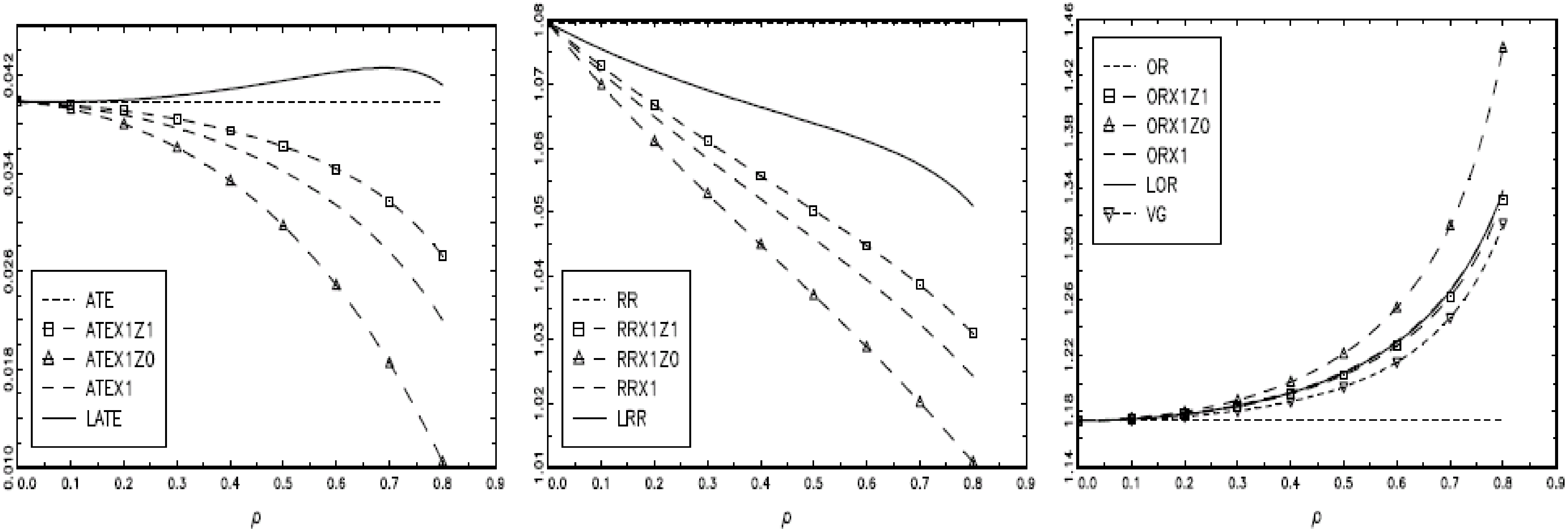

Figure 2. $\gamma = -1$, $\delta = 0.615$

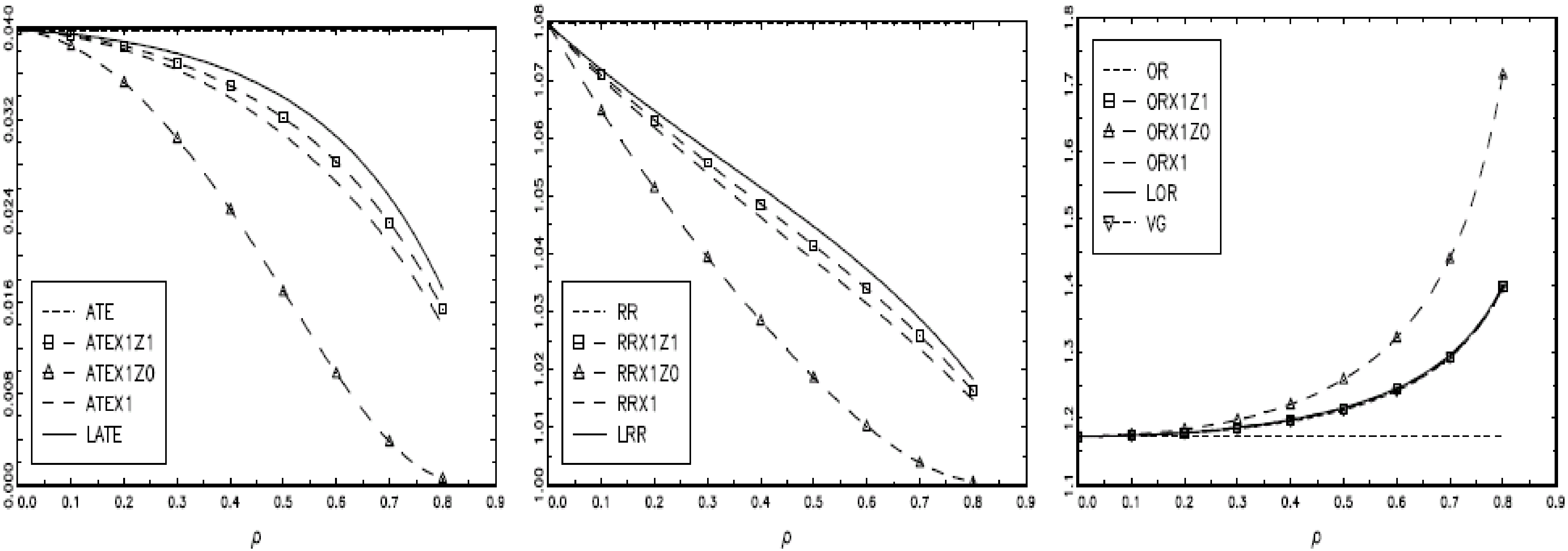

Figure 3. $\gamma = -2$, $\delta = 1.208$

Another example:

$$E\left[X \mid Z=z, U=u\right]=\operatorname{expit}\left(\alpha_1+\alpha_2 z+\alpha_3 u+\alpha_4 z u\right)$$
$$E\left[Y \mid X=x, U=u\right]=\operatorname{expit}\left(\beta_1+\beta_2 x+\beta_3 u+\beta_4 x u\right)$$

$$U \sim N\left(0,1\right)$$

$\beta_4 \neq 0$, no problem, still estimate local parameters

$a_4 \neq 0$, problem as monotonicity then doesn't hold

Table 1. Causal parameters and SMM estimates for logistic model

	$\alpha_4=\beta_4=0$		$\alpha_4=0,\beta_4=1$		$\alpha_4=1,\beta_4=1$	
	mean	*stdev*	mean	*stdev*	mean	*stdev*
ATE	0.0453	*0.0003*	0.0344	*0.0003*	0.0344	*0.0004*
ATEX1Z1	0.0455	*0.0005*	0.0625	*0.0007*	0.0708	*0.0008*
ATEX1Z0	0.0437	*0.0006*	0.0658	*0.0007*	0.0658	*0.0007*
ATEX1	0.0447	*0.0004*	0.0640	*0.0005*	0.0684	*0.0005*
LATE	0.0574	*0.0012*	0.0403	*0.0013*	0.0923	*0.0016*
SMM	0.0575	*0.0188*	0.0403	*0.0180*	0.1148	*0.0225*
RR	1.0907	*0.0006*	1.0689	*0.0007*	1.0689	*0.0007*
RRX1Z1	1.0682	*0.0009*	1.0936	*0.0011*	1.1007	*0.0012*
RRX1Z0	1.0627	*0.0008*	1.0944	*0.0011*	1.0944	*0.0011*
RRX1	1.0656	*0.0006*	1.0940	*0.0008*	1.0977	*0.0008*
LRR	1.1219	*0.0028*	1.0855	*0.0029*	1.1376	*0.0027*
SMM	1.1229	*0.0426*	1.0863	*0.0399*	1.1523	*0.0328*
OR	1.1994	*0.0014*	1.1479	*0.0016*	1.1478	*0.0016*
ORX1Z1	1.2377	*0.0032*	1.3467	*0.0045*	1.4457	*0.0057*
ORX1Z0	1.2420	*0.0034*	1.3984	*0.0053*	1.3982	*0.0052*
ORX1	1.2394	*0.0023*	1.3691	*0.0033*	1.4227	*0.0039*
LOR	1.2584	*0.0061*	1.1749	*0.0061*	1.5802	*0.0126*
SMM	1.2629	*0.0958*	1.1781	*0.0852*	2.2247	*0.3699*

Notes: 1000 Monte Carlo replications; sample size $500,000$.

Effect of BMI on Asthma, Mendelian Randomisation

ALSPAC data. Children's BMI at age 7, top quartile=1.

FTO gene causally related to BMI.

4647 observations.

Probit of BMI on FTO, coefficient = 0.13, se 0.042

Linear OLS estimator of prevalence of Asthma caused by BMI: 0.021, se 0.012.

Linear IV estimator of prevalence of Asthma caused by BMI: 0.076, se 0.27.

Multiplicative SMM estimator for risk ratio: 0.57

Puzzle?

$$E\left[Y(1) \mid X(1) > X(0)\right] = -0.10; \; E\left[Y(0) \mid X(1) > X(0)\right] = -0.18.$$